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Contributions of Multibody Dynamics to Space Flight: A Brief Review

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A brief historical review is given of developments in multibody dynamics that led to the successful prediction of attitude motions of space vehicles. The important role of methods that simplified the formulation of equations of motion of complex spacecraft is pointed out. Developments in modeling flexible bodies undergoing large overall motion are highlighted. Factors reducing computer simulation time, such as the use of symbol manipulation, recursive formulation, and parallel processing algorithms, are outlined, and real-time computation is exemplified. Applications of multibody dynamics to space flight range from solar panel deployment to servicing of satellites by the shuttle manipulator arm, construction of the space station, and more. Although the details of these examples remain unpublished in many company and agency reports, a few representative applications of multibody dynamics to spacecraft simulations are recalled.

I. Introduction

SPACE flight did not begin until about a half-century following the Wright brothers' epoch-making airplane flight in 1903. Aircraft design required manipulating the complex laws of aerodynamics and controls,[1] and although the dynamic analysis of motion involved was rather simple, motivation was lacking for a study of the dynamics of more complex, articulated multibody systems. A small number of attempts were undertaken to analyze the behavior of multibody systems such as aircraft landing gear and helicopters (for example, Refs. 2 and 3), but, for the most part, these attempts were unsuccessful because the equations governing the motions of the multibody systems under consideration were too complex to be amenable to the most advanced solution techniques of the day. In fact, though, such analyses were unnecessary because the relative simplicity of the aircraft of those years made trial-and-error engineering a viable, inexpensive option for developmental work. Thus, the inability of aircraft analysts to solve the coupled, nonlinear, differential equations of motion of even the simplest of multibody systems did not stand in the way of progress.

Soon after the start of the second half of the 20th century, however, both the necessity of analyzing multibody systems and the means to analyze them appeared nearly simultaneously. The necessity arrived on 4 October 1957 with the launching of the world's first artificial satellite, Sputnik, by the former Soviet Union. The means came with the advent of the first practical digital computers at about that same time. With the opening of the space frontier and the start of the race to be the first to land men on the moon, two old branches of dynamics, particle dynamics in the form of orbital mechanics and dynamics of rigid bodies, acquired renewed interest. The triumph of orbital mechanics in designing trajectories to the moon and devising various methods of spacecraft orbit transfer has been chronicled well by Battin.[4] Multibody dynamics, an extension of rigid-body dynamics, played an equally important part in the space age out of the operational needs of spacecraft: There were rendezvous and docking techniques to be mastered, gyroscopic control systems to be developed, solar arrays to be deployed, lunar module landing gear to be designed, and the dynamics of weightless human bodies to study. It soon became apparent that the entire subject of rigid-body dynamics needed to be reexamined, for it was rapidly proving inadequate to the tasks at hand.

Whereas the main issue in dynamics before the space/computer age was that of solving relatively simple equations of motion, the central problem in the new era was that of formulating equations of motion. The computer now made it possible to solve virtually any system of differential equations of motion one could formulate, which made the solution task less of an issue, and moved the equation formulation problem to the forefront of dynamics research. In the sequel, we trace the highlights of the developments that took place in multibody dynamics in the areas of formulation methods, numerical and symbolic computer programs used to predict motions, and in rapid simulation of spacecraft operations.

II. Early Multibody Dynamics Programs

The first paper on multibody dynamics applied to spacecraft was published in 1965 by Hooker and Margulies[5] of the Lockheed Palo Alto Research Laboratory. In it the authors developed an algorithm for formulating equations of motion of an arbitrary number of rigid bodies connected to each other in what the authors called a "topological tree," that is, in such a way that there were no closed chains of bodies in the system. Because Hooker and Margulies employed

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Newton/Euler methods in their formulation, they were confronted by the problem of how to deal with the algebraic elimination of constraint forces that appeared in the course of their derivation. As they put it,

> [o]ur derivation . . . uses Newton's equation to eliminate from the Euler equations the moments of the unknown interaction forces, i.e., the moments of the reactive forces transmitted through the joints. If there are fewer than three degrees of freedom at a joint, there will also be an unknown (attitude-dependent) constraint torque at that joint. We show how to eliminate these unknown torques in the general case. If the equations are to be integrated on a computer, it may be best to let the computer do the elimination numerically—unless the matrix of coefficients multiplying the derivatives of the angular velocity components is easily inverted algebraically. . . .

This problem was eliminated when Lagrange's method, rather than the Newton/Euler approach, was employed in the equation formulation process, but not without a price. As Hooker[6] wrote in 1970,

> . . . the Lagrangian approach suffers from the drawback that for a problem as complex as the general n-body satellite, it is exceedingly involved to derive the system kinetic and potential energies in a form suitable for subsequent differentiation in Lagrange's equations. Furthermore, in the Lagrangian formulation the equations are not written in terms of physical body axes, so it is sometimes unclear how to modify the equations—without rederiving them—to include such effects as those of additional internal forces and torques or of active control laws.

In other words, the early developers of multibody dynamics algorithms were faced with two equally undesirable choices: 1) use Newton/Euler methods, which permitted the use of variables other than generalized coordinates, such as angular velocity measure numbers, that led to relatively simple equations of motion, but necessitated the algebraic elimination of constraint forces and constraint torques that did not contribute to the equations of motion, or 2) use Lagrangian methods, which automatically eliminated noncontributing constraint forces and constraint torques, but forced one to use variables and procedures that led with excessive labor to unnecessarily complex equations. There were other equation formulation methods, in addition to Newton/Euler and Lagrange, that were available to analysts at the dawn of the space age, such as D'Alembert's principle, Hamilton's method, the Boltzmann/Hamel method, and Gibbs–Appell method, but all of these came with their own specific drawbacks (see Kane and Levinson[7]).

From the point of view of efficiency of formulation that was needed for the equations of motion of complex spacecraft systems, several things were necessary. Obviously, a method had to be developed that did not suffer from the shortcomings of any of the previous methods. That is, the method would have to permit the user to employ the widest possible choice of independent variables, unlike Lagrange's and Hamilton's methods; it would have to be operationally straightforward, unlike virtual work methods; it would have to eliminate automatically noncontributing force and torque measure numbers from the equations of motion, unlike the Newton/Euler method; it would need to circumvent the differentiation of lengthy scalar functions, unlike the Lagrange, Hamilton, Boltzmann–Hamel, and Gibbs–Appell methods; and it would have to lead directly to the simplest possible equations, unlike the Lagrange, Hamilton, and Boltzmann–Hamel methods. Kane's method of formulating equations of motion, a method that satisfied all of these requirements, originated with the space age and is described in the next section.

III. Dynamic Equation Formulation Made Simple: Kane's Method

Before the multibody era, an analyst's choice of method for formulating equations of motion was nothing more than a matter of personal taste because the equations of motion for the simple systems being analyzed could be derived equally well using just about any method of classical mechanics. However, to the dismay of many analysts, when they attempted to apply these methods to a complex, multibody system, the previously unnoticed shortcomings of the methods became immediately apparent. It was discovered that a given method either forced one to perform so much labor to obtain equations of motion, or led to equations of such great complexity, that use of the method became prohibitive.[7]

In 1960, Kane[8–10] published the first in a series of three papers that set forth the basic principles of a new, systematic method of formulating equations of motion. The method is based on the use of new kinematic quantities discovered by Kane, today called partial angular velocities and partial velocities, that revolutionized the equation formulation process. Note that Kane's method was introduced during the computer era with special attention given to operational ease. By simply dot multiplying partial angular velocities and partial velocities with readily available vector quantities, such as inertia torques, inertia forces, contact torques, and contact forces, an analyst could generate equations of motion for any system. These dot products automatically eliminated noncontributing forces and torques from the equations of motion without necessitating the use of algebraic means, as in the Newton/Euler method, and made it unnecessary to differentiate either the massive kinetic energy functions of Lagrange's method or the Gibbs functions of Gibbs–Appell method. Note that Jourdain[11] in 1909 had proposed dot multiplying inertia forces and active forces with variational velocities to derive the equations of motion for a system of particles, but he did not elaborate on the crucial details as to how to construct these variations. In essence, Jourdains's variational velocities are the same as Kane's partial velocities multiplied by variational derivatives of the generalized coordinates; however, that shows that one does not need the variational derivatives in the first place. Actually Kane went further than Jourdain[11] by introducing partial angular velocities to take care of rigid bodies and using quasi coordinates rather than generalized coordinates to formulate the equations of motion. Kane's remarkable discovery was that the partial velocities and partial angular velocities are significant in their own right, in the ubiquitous role they play in all of dynamics such as treating problems of impact, bringing workless constraint forces into evidence, and forming of linearized equations without developing the nonlinear equations (see Kane and Levinson[12]). Further, Kane showed with his use of generalized speeds as the motion variables that the final equations are simplified with much less labor than is required in connection with the Boltzman–Hamel method, which used similar variables. Perhaps the most practical feature of Kane's method was that it provided a clear, operational way of determining generalized active forces. This is in striking contrast to virtual work methods, which become nonoperational for rigid bodies because variations in finite rotations are not defined because finite rotations are not vectors. Indeed, it is only by using partial angular velocities and partial velocities, whether one calls them by these names or not, that one can make virtual work methods operational. Another advantage of Kane's method is that it appears directly to nonholonomic systems. A comprehensive discussion of advantages of Kane's method in connection with multibody dynamics is given by Huston.[13]

IV. Multibody Dynamics and Computerized Symbol Manipulation

In the 1960s, people working in the field of orbital mechanics began using computers to obtain series expansions in literal form. This use of computers to perform literal rather than numerical analyses came to be called computerized symbol manipulation. Early attempts to apply computerized symbol manipulation to the formulation of explicit equations of motion were not successful largely because of the problem of intermediate-swell. For example, when Lagrange's method was employed, one first had to use the symbol manipulation program to formulate the kinetic energy of the system under consideration, a quantity often of no interest in its own right. Then one had to differentiate this kinetic energy function with respect to the time derivative of the first generalized coordinate, a process that generated another expression that was of no intrinsic interest. This expression next had to be differentiated with respect to time, which necessarily produced a very large expression that had to be stored in computer memory while the program differentiated the kinetic energy with respect to the first generalized coordinate and then subtracted the resulting expression from the stored one. As was often the case, these intermediate expressions took up more memory than did the final result. Although the computer often had enough memory to store the final result, it all too often did not possess enough memory to store the intermediate expressions needed to produce the final result. Consequently, no final result could be produced in practice. Intermediate-swell was, thus, a major issue in early attempts to employ symbol manipulation in multibody dynamics.

Newton/Euler and Kane's methods are particularly suitable for symbol manipulation. Because they do not involve the multiple differentiations of energy functions, but instead employ dot products of vectors, the intermediate-swell problem is a nonissue. Furthermore, they allow one to pick quantities other than generalized coordinates as dependent variables, which further reduced memory requirements. A demonstration of the feasibility of combining Kane's method with symbol manipulation was made by Levinson.[14] About the same time Schiehlen and Kreuzer[15] announced the development of a multibody dynamics code, NEWEUL, based on symbol manipulation applied to Newton/Euler equations. Not long after that, Rosenthal and Sherman[16] developed a commercial program called SD/EXACT, which was based on Kane's method. They state[16]

> [t]he need to simulate motions of complicated aerospace vehicles has led to widespread development of practical multibody simulation codes. Most currently available programs are, however, difficult to use and expensive to run. In this paper we describe a new approach to multibody simulations offering significant improvements in run time performance and usability. The use of Kane's formulation for the equations of motion, together with symbolic equation manipulation techniques, is shown to provide a powerful method for treating multibody problems. A multibody program, SD/EXACT, based on these ideas has been implemented and is described. We . . . show by direct measurement that SD/EXACT can produce simulations whose performance is comparable to the best hand-developed simulations, while conventional multibody programs can lead to simulations with execution times as much as an order of magnitude higher.

SD/EXACT has been employed in industry on various flight-related projects, including the Galileo spacecraft.

In recent years, other symbolic dynamics programs have been developed, such as AUTOLEV[17] and SYMKIN,[18] where the latter uses closed-form solutions with the aid of Mathematica wherever possible. AUTOLEV, which is based on Kane's method, provides the user with a significant amount of control over the details of the equation-formulation process. Typically, symbol manipulation codes such as AUTOLEV are suitable for a system of a few rigid bodies, which is surprisingly adequate for making basic models for most spacecraft. Botz[19] has used AUTOLEV to generate flexible-body models, but the modeling of multiple flexible bodies and the handling of associated finite element data remains a challenge. The Appendix shows an AUTOLEV command file written by David Levinson at Lockheed Martin Advanced Technology Center for simulating the spinup motion of a cantilever beam, the details of which are described in the next section. The advantage of making changes in the dynamic model and having a FORTRAN or C code ready to run and plot results, all in a short time, can hardly be overstated.

V. Dynamics of Multibody Systems with Flexible Components

Most commercially available multibody codes that can handle a large number of bodies and particularly those modeling flexible-body dynamics continue to be numerical. By the end of the first decade of the space age, it had become clear that not all spacecraft were amenable to analysis by multibody programs that only considered rigid bodies. That this was the case first became apparent shortly after the launching of America's first satellite, Explorer I, on 31 January 1958. Explorer I was a pencil-shaped object, injected into orbit spinning about its long axis. According to rigid-body dynamics, spin about this minimum axis of inertia should have been a stable motion. However, within a single 90-min orbit the long axis of the satellite was precessing on the surface of a cone having a 60-deg half-angle, an instability subsequently attributed by Bracewell and Garriott[20] to the energy-dissipative, flexural motions of four radial wire antennas that were attached to Explorer I, which had been ignored in all prelaunch analyses.

Soon thereafter, analysts throughout the aerospace industry began producing multibody motion simulation programs that could accommodate flexible bodies. The big breakthrough in analysis of flexible bodies had, of course, come from the discovery of the finite element method, which represents one of the great triumphs of engineering science in the second-half of the 20th century and may have started with the pioneering paper of Turner et al.[21] The method could analyze any elastic structure with complicated boundary conditions and soon developed into a vast field. Multibody dynamicists proceeded by taking advantage of the fact that, as Likins[22] put it,

> . . . idealizing a spacecraft as a collection of interconnected rigid bodies to some of which are attached linearly elastic flexible appendages leads to equations of motion expressed in terms of a combination of discrete coordinates describing the arbitrary rotational motions of the rigid bodies and distributed or modal coordinates describing the small, time-varying deformations of the appendages. . . .

In using modes to describe deformations of the elastic components of a multibody system, two questions have to be answered in connection with each component: What type of modes should one use and how many modes should one retain in the model? Here the objective is to choose component modes for the flexible components such that the vibrational motion of the assembled system is the same as that of the entire structure analyzed as a whole. This is a necessity in the vibrational analysis of aircraft and rockets, where the different parts of the aircraft, such as the wings and the fuselage, or the different stages of the rocket, often are manufactured and analyzed in different companies, and then a separate analysis is done to determine the system modes of the assembled structure. This problem of choosing component modes to account properly for conditions at the boundaries between components, together with component mode synthesis to obtain the system modes, was solved in 1965 by Hurty.[23] A slightly different method, due to Craig and Bampton,[24] has found widespread use in the aerospace industry.

Procedures for determining how many modes to keep fall under the heading of model reduction. In the context of flexible structures, the idea is to select modes that can be excited by the forces present and also can contribute to responses of interest. This often means that simply using the first few modes may not be a good choice. Hughes and Skelton[25] gave several modal truncation criteria at the appendage and the system level, based on a modal completeness index[26] as well as modal cost.[27] Spanos and Tsuha[28] applied balanced realization with different choices of component modes at the component level and then checked the results at the system level. Lee and Tsuha[29] selected the modes at the system level and then "decomposed" the system modes into component modes using static correction modes to augment component or system modes. In both Refs. 28 and 29, the Galileo spacecraft was used (Fig. 1), modeled with two flexible components, a rotating antenna and a stator, and high-fidelity reduced-order models were shown that were then used for the nonlinear simulation, using the multibody dynamics software, DISCOS.[30]

Representing deformations of flexible appendages by their component modes had become a mainstay of almost all programs for simulating motions of multibody systems with flexible components, and for about two decades, there was no reason to suspect that anything could be wrong with this approach. Then, in 1987, Kane et al.[31] discovered a subtle flaw in the fundamental theory underlying these programs that caused an erroneous dynamic softening for rotating beams. Actually the phenomenon leading to the error was well known[32] in the rotorcraft industry and is best described with a specific example, as follows, commonly referred to as the spinup problem in many papers written on this subject since the publication of Ref. 31.

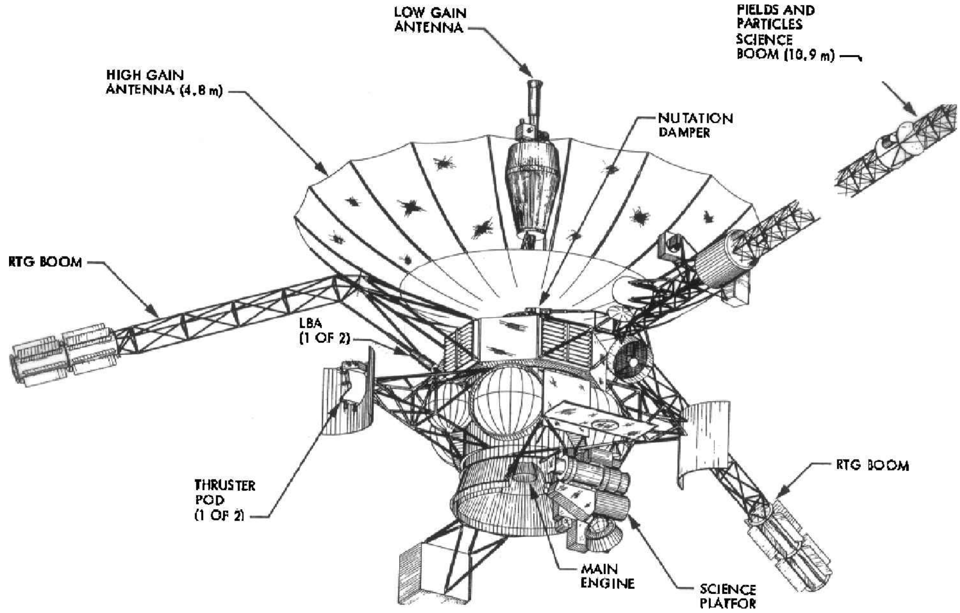

Fig. 1 Galileo spacecraft built by the Jet Propulsion Laboratory.

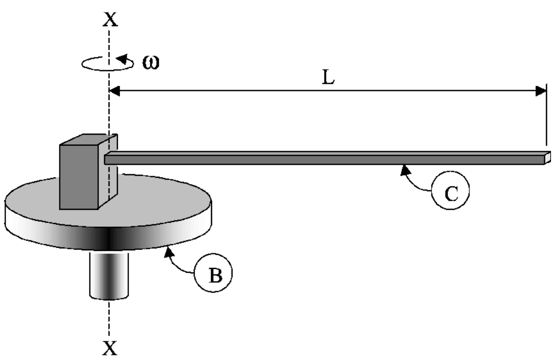

Fig. 2 Cantilever beam attached to a rotating base.

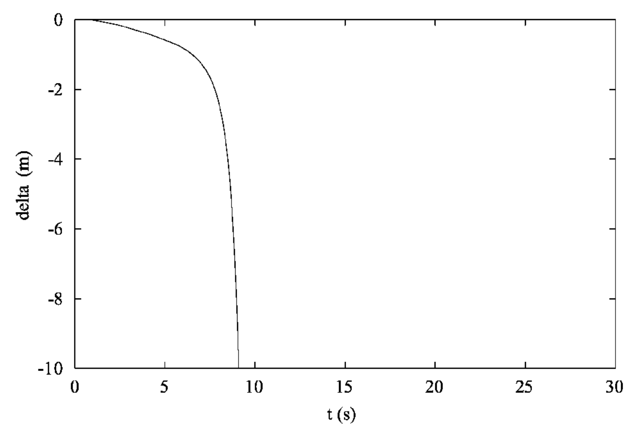

Fig. 3 Unbounded beam tip motion predicted by multiflexible-body programs not incorporating geometric stiffness due to inertia forces.

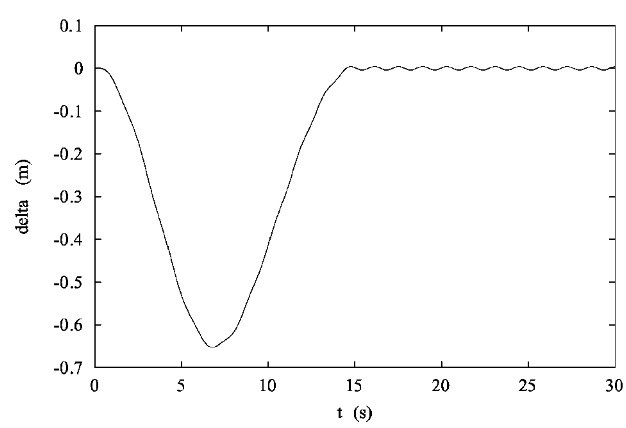

Fig. 4 Correct beam tip motion predicted by multiflexible-body programs incorporating geometric stiffness due to inertia forces.

Figure 2 shows a uniform cantilever beam C supported by a base B that can be made to rotate about a fixed vertical axis X with X passing through one end of C. At time $t = 0$, the system is at rest, and C is undeformed. B then is made to rotate in such a way that ω, the angular speed of B (in radians per second), is given by

$$\omega = \begin{cases} \frac{2}{5}[t - (7.5/\pi)\sin(\pi t/7.5)] & 0 \le t < 15 \\ 6 & 15 \le t \le 30 \end{cases} \qquad (1)$$

C has length L, Young's modulus E, mass per unit of length ρ, and centroidal moment of inertia I of the cross-sectional area. Let δ denote the displacement of the free end of C, in the plane of rotation, relative to a line fixed in B and coincident with the centroidal axis of C when C is undeformed. With gravity neglected and with $L = 10.0$ m, $E = 7.0 \times 10^{10}$ N/m^2, $\rho = 1.2$ kg/m, and $I = 2.0 \times 10^{-7}$ m^4, plot δ as a function of t for $0 \le t \le 30$ s.

It was found to the dismay of many dynamicists that when they employed their heretofore reliable general-purpose multi-flexible-body simulation programs, using finite element modal data to represent beam flexibility, they all obtained the plot shown in Fig. 3, which predicted unbounded beam tip motion. As shown in Ref. 31, the correct response (Fig. 4) could be produced only after the analysts retained, up to a certain stage in the analysis, nonlinearities that were specific only to beams. If the beam were replaced with another structure, for example, a plate, then, as shown by Banerjee and Kane,[33] only retention of nonlinearities, specific to the deformation of a plate, up to a certain stage in the analysis could correct the simulation. It suddenly seemed that the generality obtainable with multi-rigid-body programs would be impossible to achieve with multi-flexible-body programs.

What was the source of the error and why was fixing it so problematical? The error was a direct result of what had become standard procedure in formulating equations of motion for systems containing flexible bodies: Elastic deformations, assumed small, were described in terms of vibration modes multiplied by modal coordinates, but vibration modes are based on linear elasticity theory, which neglects any geometric coupling between axial and transverse displacements. Use of vibration modes with inherent neglect of this geometric nonlinearity amounted to a premature linearization, with unwarranted dropping of first-degree terms and loss of stiffness. Why was this? It was because the linearization could be delayed until the appropriate stage in the analysis only when the specific nature of the structure, that is, whether it was a beam, a plate, etc., being analyzed was known a priori in all detail. If the general nature of the structure was to be maintained in the algorithm, which was the basic idea of a multibody program in the first place, then the analyst had no choice but to perform the linearization prematurely.

Much time and money had gone into the development of these multi-flexible-body programs, and these revelations caused some discord briefly in the aerospace industry. A hastily organized meeting[34] in Pasadena in October of 1987, originally intended to deal in a professional manner with this single purely technical issue, soon was transmogrified into something entirely different. Claims were made that, because the premature linearization led to unbounded motion predictions only when the spacecraft rotations were performed sufficiently rapidly, that the flaw really was not a flaw, but merely an engineering approximation. Others pointed out that one really could not trust any simulation performed by one of these multi-flexible-body programs because it was not possible to know a priori at what angular speed the incorrect predictions ended and the correct predictions began. Clearly what was needed was a technical solution to a technical problem.

The solution came independently from Banerjee and Dickens[35] and Wallrapp and Schwertassek.[36] Their procedure entailed formulating equations of motion for generic flexible bodies in precisely the same manner as before, premature linearization and all, and then adding to the necessarily flawed equations geometric stiffness due to operating loads. In Ref. 35, it was found that 12 inertia loads represent the most general reference frame motion, and their use reproduced the results of special-purpose programs for rotating beams and plates. This also made it possible to repair existing multibody codes with relatively little effort. As a result, proprietary codes, such as Lockheed's DYNACON,[37] and public-domain software, such as SIMPACK,[38] now incorporate geometric stiffness terms. General-purpose multibody programs such as these, unlike their predecessors, correctly produce the plot shown in Fig. 4 for the beam spinup problem, rather than the incorrect plot displayed in Fig. 3, without the need for ad hoc modifications. An example of dynamic softening due to inertia loads occurs in elastic rockets, as shown by Banerjee,[39] where Kane's equations were extended to the dynamics of variable mass elastic bodies.

Examples of aerospace applications of dynamics of flexible bodies undergoing large overall motions include such systems as high-performance aircraft[40] and the shuttle-borne tethered satellite.[41] Some of the more well-known examples of flexible articulated multibody dynamics have been the shuttle manipulator arm grabbing a satellite for servicing and the operations of the International Space Station, a gigantic structure as large as a football field with an articulated main truss to which solar panels, thermal radiators, experimental modules are attached. Modi et al.[42] present a simulation of the dynamics of the space station that carries a mobile servicing construction unit that is essentially a two-arm manipulator with a dexterous end effector.

Finally, note that for highly flexible space structures there is no guarantee that deformations will remain "small" in accordance with linear structural theory, and under these circumstances, modal methods are inapplicable. For example, Vu-Quoc and Simo[43] and Shabana et al.[44] consider cases that employ geometrically exact nonlinear strain theories applicable to systems undergoing large overall motions that include large elastic deformations. In an alternative approach, Banerjee and Nagarajan[45] matched predictions of nonlinear finite element theory for large deformation and large overall motion with a model made of many rigid elements connected by linear springs with stiffness coefficients derived from linear beam theory.

VI. Making Multibody Dynamics Simulations Run Faster

Computer simulations of motions of flexible spacecraft, modeled either by the use of modes or as systems of rigid bodies interconnected by springs tend to be time consuming. It has been found that several factors determine the computer time required to perform a simulation of the motion of a multibody system, namely, the equation formulation method employed, the choice of variables used in the equations, whether the equations are generated numerically or symbolically, and the method used to integrate numerically the equations of motion.

As has already been pointed out in Sec. III, use of Kane's method leads to the simplest possible equations of motion with the least amount of labor (also see Ref. 7). With regard to the choice of variables, it is well known that time derivatives of quasi coordinates, or generalized speeds[12] as Kane calls them, generally lead to simpler equations than those produced with generalized coordinates as dependent variables. DISCOS[30] may have been the first public domain multibody software that used quasi coordinates; later, Meirovitch[46] formalized their use in the context of Lagrange's equations. Mitiguy and Kane[47] have shown that a certain choice of generalized speeds associated with revolute joints produces significant computational savings, whereas the work of D'Eleuterio and Barfoot[48] on special variables for describing elastic motion holds forth the promise of improved computational efficiency.

In connection with numerical solutions, Orlandea et al.,[49] in establishing the theory for the commercially available code ADAMS, found that simulation time is significantly reduced when one employs uses sparse matrix techniques along with a DISCOS-type formulation with redundant coordinates and motion constraints, together with a backward difference scheme for the integration of the resulting system of differential-algebraic equations.[50]

Another powerful technique, developed over the past two decades, for improving the speed of multibody simulations is the so-called order-n algorithm. This work grew out of the observation that all equation formulation methods employing the minimum number n of motion variables give rise to a dense $n \times n$ matrix of coefficients of time derivatives of the motion variables, the decomposition and backsolving of which requires on the order of n^3 arithmetic operations. Armstrong,[51] using the shuttle manipulator arm as an example, showed that the mass matrix could be diagonalized, thus requiring order n, rather than order n^3 operations for decomposition, provided that the dynamic equations were derived in a recursive manner. Bae and Haug[52] used the virtual work approach in a recursive formulation embedded in the commercial multibody dynamics code DADS to develop the equations of motion of a system of rigid bodies. Rosenthal,[53] using Kane's method, showed that, although the standard, order-n^3 formulation is better than the order-n formulation for a six-degree-of-freedom robot, the order-n method outperforms the standard method for systems having more than 10 degrees of freedom, with vast improvement for even larger systems. The order-n method is particularly desirable when a multibody system with flexible components is described by many modes, as illustrated by the order-n version of the code TREETOPS,[54] originally developed using Kane's method. Rosenthal's recursive algorithm[53] has been extended in Ref. 37 in block-diagonal mass-matrix form to systems containing flexible bodies with geometric stiffness and motion constraints. Jain and Rodriguez[55] developed an "innovations operator factorization" of the mass matrix, borrowing a technique from Kalman filtering, for a highly efficient flexible-multibody dynamics code DARTS that was used to simulate the Cassini spacecraft. An order-n formulation via Lagrange's equations has also been developed by Pradhan et al.[56] for flexible-body systems and was used in simulations of the space station.

Parallel processing algorithms with operation counts of order $\log_2 n$ have been developed for systems of rigid bodies by Fijani et al.,[57] who showed that the order-n^3 formulation employing redundant coordinates, motion constraint equations, and Lagrange multipliers is actually computationally faster than an order-n formulation that uses the minimum number of dependent variables. A quasi-order $\log_2 n$ algorithm using $\mathcal{O}(n)$ number of processors was developed by Anderson and Duan[58] by cutting a system into subsystems whose equations are evaluated in parallel before enforcing the system constraints. Featherstone using a similar "divide and conquer'[59,60] strategy produced the fastest $\mathcal{O}(\log_2 n)$ algorithm for a large number of bodies and processors by recursive application of a formula that constructs the equations of motion of the system from those of its constituent parts.

Modern multibody dynamics numerical codes mentioned in this and the preceding sections come with menu-driven user interfaces and on-screen graphics, so that an engineer can be an effective user of such a program without being an expert in dynamics. This explains the widespread use of multibody codes in the aircraft, rocket, and spacecraft industries. The simulations by ADAMS of the Mars Pathfinder, Cassini spacecraft, space shuttle, and Space Station *Alpha* are typical examples. Finally, note that the reduced computation advantage provided by symbolic-manipulation-based programs and the runtime versatility of numerical order-n codes have been achieved by modern, object-oriented C^{++} coding; for example, see Otter et al.[61] For recent comprehensive surveys of multibody dynamics codes, see Schiehlen[62] for systems of rigid bodies and Shabana[63] for systems with flexible components.

VII. Real-Time Simulations

Multibody dynamics simulations performed in real time have been used not only to check spacecraft operations on ground, but also in situations such as training of astronauts. The systems engineering simulator at NASA Johnson Space Center, for example, performs real-time simulation of the remote manipulator systems for the space shuttle and the space station. The process is best described in terms of an example such as the space station remote manipulator system (SSRMS), which has eight links arranged in a chain with seven active joints, shown in Fig. 5. Typically, one end of the SSRMS latches on to the space station, while the other end grapples a payload. The manipulator can impart six degrees of freedom to its payload and has an extra degree of freedom to overcome singular situations in the control of its motion. The dynamics of the SSRMS is modeled as a multibody system with 10 flexible bodies representing the space station, 8 links of the SSRMS, and the payload. External forces (reaction control jets, control moment gyro torques, aerodynamics forces, etc.) are applied to the base and the payload, and torques from joint servomotors are applied at the joints. The equations of motion of the system is first written in the partitioned mass matrix form

$$\begin{bmatrix} A_{RR} & A_{RF} \\ A_{FR} & A_{FF} \end{bmatrix} \begin{bmatrix} \ddot{X}_R \\ \ddot{X}_F \end{bmatrix} + \begin{bmatrix} 0 \\ K_F X_F \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ \tau \end{bmatrix} \\ 0 \end{bmatrix} + \begin{bmatrix} G_R \\ G_F \end{bmatrix} + \begin{bmatrix} N_R \\ N_F \end{bmatrix} \tag{2}$$

where, the subscripts R and F represent quantities associated with rigid and flexible coordinates of the system, respectively. A, X, G, and N are the mass matrix, generalized coordinates, generalized forces due to external forces, and nonlinear part of generalized inertia forces, respectively. X_R represents 13 rigid-body coordinates, of which 6 are for the rigid degrees of freedom of the base body and 7 are for the seven driven joints. X_F represents the component modal coordinates of the system, obtained by collecting all the modal coordinates of the 10 bodies. K_F is the generalized stiffness matrix of the system with all rigid degrees of freedom locked, and τ is the seven-element column matrix containing the seven joint torques. These torques represent servomotor action and joint friction.

For real-time simulation, the base body and payload are treated as rigid, and 28 modal coordinates represent X_F to start with. The dimension of the model in Eq. (2) is then reduced by modal truncation of the system at the instantaneous configuration with all of its rigid degrees of freedom locked. The vibration equation of this configuration is given by

$$A_{FF}\ddot{X}_F + K_F X_F = 0 \tag{3}$$

The modal transformation matrix Ψ is chosen such that

$$\Psi^T A_{FF} \Psi = I \text{ (unit matrix)}, \qquad \Psi^T K_F \Psi = \Lambda \text{ (diagonal matrix)} \tag{4}$$

Modal truncation by retaining modal coordinates corresponding to only a desired number of lowest frequencies of the system leads to

$$X_F = \Psi_r \eta_r \tag{5}$$

When Eqs. (4) and (5) are used in Eq. (2), the reduced equation of the system is obtained as

$$\begin{bmatrix} A_{RR} & A_{Rf} \\ A_{fR} & \Lambda_r \end{bmatrix} \begin{bmatrix} \ddot{X}_R \\ \ddot{\eta}_r \end{bmatrix} + \begin{bmatrix} 0 \\ \eta_r \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ \tau \end{bmatrix} \\ 0 \end{bmatrix} + \begin{bmatrix} G_R \\ G_f \end{bmatrix} + \begin{bmatrix} N_R \\ N_f \end{bmatrix} \tag{6}$$

where

$$A_{Rf} = A_{RF}\Psi_r, \qquad A_{fR} = \Psi_r^T A_{RF}$$
$$G_f = \Psi_r^T G_F, \qquad N_f = \Psi_r^T N_F \tag{7}$$

Typically, only 10 retained modes have been found to be sufficient for real-time applications. The SSRMS is moved very slowly to allow full control of its motion, and advantage is taken of this slow motion in that all products of structural deflections, their rates, and rigid-body rates are ignored. Nonlinear terms of rigid rates are, however, retained.

Fig. 5 Space station manipulator arm.

The key to the real-time implementation of the model lies in computing different variables of Eq. (6) at different time intervals and using parallel processing. Because the system configuration changes very slowly, the computation of the mass matrix A, the modal transformation matrix Ψ_r, and N are performed at intervals of 80 ms (called slow interval). Because Ψ_r is modified at the end of every slow interval, the modal variable η_r and its rates are reinitialized at the beginning of every such interval by using a reverse transformation, from the latest value of X_F, so that timewise continuity of X_F is maintained. The generalized force term G is updated at 5-ms intervals. However, some components, such as the station control forces and moments, which are updated at lower frequencies, are computed at frequencies corresponding to the actual hardware. The joint torque τ is generated at 0.5 ms. Equation (6) is solved for the highest derivatives and integrated to update the generalized coordinates and their rates at 0.5 ms. The model for the joint control system that produces joint rate commands for the joint motors is executed at 50 ms using the actual onboard software, on a parallel processor. The model scenarios and the models for control system for the space station are also run on parallel processors.

VIII. Future Contributions of Multibody Dynamics to Spacecraft Simulation

At present, there are several categories of spacecraft problems that have not yet been dealt with adequately and may be the focus of future research in multibody dynamics. One of these is the prediction of motions of spacecraft carrying arbitrarily shaped, partially filled fuel tanks in which liquid fuel is free to slosh. So far, no general methodology has been established whereby fuel sloshing can be accommodated in a multibody program with the same ease as are flexible components undergoing small vibrations. Modeling sloshing fuel crudely as collections of oscillating rigid pendula or oscillators[64] currently is a common approach to this problem in the aerospace industry, and so there is much room for improvement in this area. Finite element modeling of contact conditions in multibody dynamics, as has been done by Amirouche et al.,[65] is another area requiring understanding how micromechanical phenomena of impact influences macromechanical multibody motion. Another topic of concern is the dynamics of large, inflatable flight vehicles.[66] Such objects are capable of undergoing large deformations, possess very little structural stiffness, and do not fall under the purview of conventional modal analysis of structures. Future multibody programs will be expected to analyze correctly inflatable structures in a generic way.

IX. Conclusion

Multibody dynamics has played a central role in space flight by providing high fidelity simulations of spacecraft operational tasks ranging from solar panel and antenna deployment to assembly of the space station by a robotic manipulator. This capability of predicting the dynamic behavior of spacecraft at the design and analysis stage has been essential to the success of the actual flight. New developments in formulating equations of motion, modeling of flexible bodies, efficient algorithms, and computation capabilities have been motivated by the demands of spacecraft operations and, in return, they constitute the overall contribution of multibody dynamics to space flight.

Appendix: Autolev Input Script for Simulating the Spinup of a Cantilever Beam

```
% SPINUP1.AL
AUTORHS ALL
FACTORING OFF
OVERWRITE ON
AUTOEPSILON 1.0E-10
NEWTONIAN N
BODIES A
PARTICLES P
POINTS Q,AP
VARIABLES U{3}',Q{3}',DX
CONSTANTS PHI{3},R,L,EI,RHO,X,ON_OFF
SPECIFIED Y'',OMEGA'
AUTOTAYLOR(0:1,U1,U2,U3,Q1,Q2,Q3)
V_AO_N>=0>
W_A_N>=OMEGA*A3>
Y=PHI1*Q1+PHI2*Q2+PHI3*Q3
Q1'=U1
Q2'=U2
Q3'=U3
P_AO_Q>=R*A1>
V2PTS(N,A,AO,Q)
P_Q_AP>=X*A1>+Y*A2>
V2PTS(N,A,Q,AP)
V_P_A>=DT(RHS(Y))*A2>
V1PT(N,A,AP,P)
ALF_A_N>=DT(W_A_N>,N)
A_AO_N>=DT(V_AO_N>,N)
A_P_N>=DT(V_P_N>,N)
MASS A=MA,P=RHO*DX
FSTAR=FRSTAR()
EXPAND(FSTAR,1:2)
SPECIFIED MB,EB,IB,E{3},F{3},G{3,3},H{3,3},A{3,3}
FSTAR=REPLACE(FSTAR,PHI1^2=G11/(RHO*DX),PHI2^2=G22/(RHO*DX), &
                    PHI3^2=G33/(RHO*DX),PHI1*PHI2=G12/(RHO*DX), &
                    PHI1*PHI3=G13/(RHO*DX),PHI2*PHI3=G23/(RHO*DX))
FSTAR=REPLACE(FSTAR,PHI1*RHO*X=F1/DX,PHI2*RHO*X=F2/DX,PHI3*RHO*X=F3/DX)
FSTAR=REPLACE(FSTAR,PHI1=E1/(RHO*DX),PHI2=E2/(RHO*DX),PHI3=E3/(RHO*DX))
FSTAR=REPLACE(FSTAR,X^2=IB/(RHO*DX))
FSTAR=REPLACE(FSTAR,X=EB/(RHO*DX))
FSTAR=REPLACE(FSTAR,RHO=MB/DX)
```

```
F_ELASTIC[1]=-(H11*Q1+H21*Q2+H31*Q3)
F_ELASTIC[2]=-(H12*Q1+H22*Q2+H32*Q3)
F_ELASTIC[3]=-(H13*Q1+H23*Q2+H33*Q3)
F_GEOMETRIC[1]=-(A11*Q1+A21*Q2+A31*Q3)*OMEGA^2*ON_OFF
F_GEOMETRIC[2]=-(A12*Q1+A22*Q2+A32*Q3)*OMEGA^2*ON_OFF
F_GEOMETRIC[3]=-(A13*Q1+A23*Q2+A33*Q3)*OMEGA^2*ON_OFF
ZERO=FSTAR+F_ELASTIC+F_GEOMETRIC
CONSTANTS LAMBDA{3}
MB=RHO*L
E1=2*MB*(1+EXP(-2*LAMBDA1)+2*EXP(-LAMBDA1)*COS(LAMBDA1))/ &
   (LAMBDA1*(1-EXP(-2*LAMBDA1)+2*EXP(-LAMBDA1)*SIN(LAMBDA1)))
E2=2*MB*(1+EXP(-2*LAMBDA2)+2*EXP(-LAMBDA2)*COS(LAMBDA2))/ &
   (LAMBDA2*(1-EXP(-2*LAMBDA2)+2*EXP(-LAMBDA2)*SIN(LAMBDA2)))
E3=2*MB*(1+EXP(-2*LAMBDA3)+2*EXP(-LAMBDA3)*COS(LAMBDA3))/ &
   (LAMBDA3*(1-EXP(-2*LAMBDA3)+2*EXP(-LAMBDA3)*SIN(LAMBDA3)))
F1=2*MB*L/LAMBDA1^2
F2=2*MB*L/LAMBDA2^2
F3=2*MB*L/LAMBDA3^2
G11=MB
G22=MB
G33=MB
G12=0
G13=0
G23=0
H11=LAMBDA1^4*EI/L^3
H22=LAMBDA2^4*EI/L^3
H33=LAMBDA3^4*EI/L^3
H12=0
H13=0
H23=0
A11= 1.193336374E+00*MB
A12=-6.858552834E-01*MB
A13=-7.923792208E-01*MB
A21=-6.858552834E-01*MB
A22= 6.478224863E+00*MB
A23= 1.694078848E-01*MB
A31=-7.923792208E-01*MB
A32= 1.694078848E-01*MB
A33= 1.785951988E+01*MB
PHI1_L=2*((1+EXP(-2*LAMBDA1))*SIN(LAMBDA1)-COS(LAMBDA1) &
         *(1-EXP(-2*LAMBDA1)))/ &
          (1-EXP(-2*LAMBDA1)+2*EXP(-LAMBDA1)*SIN(LAMBDA1))
PHI2_L=2*((1+EXP(-2*LAMBDA2))*SIN(LAMBDA2)-COS(LAMBDA2) &
         *(1-EXP(-2*LAMBDA2)))/ &
          (1-EXP(-2*LAMBDA2)+2*EXP(-LAMBDA2)*SIN(LAMBDA2))
PHI3_L=2*((1+EXP(-2*LAMBDA3))*SIN(LAMBDA3)-COS(LAMBDA3) &
         *(1-EXP(-2*LAMBDA3)))/ &
          (1-EXP(-2*LAMBDA3)+2*EXP(-LAMBDA3)*SIN(LAMBDA3))
DELTA=EVALUATE(Y,PHI1=PHI1_L,PHI2=PHI2_L,PHI3=PHI3_L)
OMEGA=SPLINE(TRANSITION,T,0,15,0,6)
INPUT L=10,R=1,EI=14E3,RHO=1.2
INPUT LAMBDA1=1.8751040687120
INPUT LAMBDA2=4.6940911329742
INPUT LAMBDA3=7.8547574382376
INPUT U1=0,U2=0,U3=0,Q1=0,Q2=0,Q3=0
INPUT TFINAL=30,INTEGSTP=.01
INPUT ON_OFF=1
UNITS T=S,[R,L,DELTA]=M,EI=N*M^2,OMEGA=RAD/S
UNITS RHO=KG/M,[LAMBDA1,LAMBDA2,LAMBDA3,ON_OFF]=NO_UNITS
UNITS [Q1,Q2,Q3]=M,[U1,U2,U3]=M/S
OUTPUT T,OMEGA,DELTA
CODE DYNAMICS() SPINUP1.FOR,SUBS
```


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